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Buoyancy-surface tension instability by the method of energy

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The energy theory, giving a sufficient condition for stability, is developed for the motions in a horizontal, heated layer subject to buoyancy and surface tension effects. The free surface is assumed to be non-deformable (Pearson's 1958 model).

It is shown that the equations governing the energy theory are the symmetric part of the time-independent linear theory problem, and that the surface tension terms *behave* like a bounded perturbation to the Bénard problem. The qualitative behaviour of the optimal stability boundary as a function of its parameters is given. The optimal stability boundary is computed, and compared with previous linear and non-linear stability theories in terms of allowable subcritical instabilities.

1. Introduction

The recent work of Joseph and his co-workers has convincingly shown that the methods of energy and linear stability theory complement each other in demarking that region of parameter space, in which subcritical instabilities are allowable. The former limit gives a sufficient condition for stability, the latter a sufficient condition for instability. Those instabilities due to the finite amplitude of disturbances are allowable only between the two limits.

In the case of Bénard convection subject to the Boussinesq approximation, the time-independent linear operator of the governing equations is self-adjoint, and the only non-linearities are convective in nature. As a result, the critical values R_E and R_L of the Rayleigh number given by the energy and linear theories are identical (Joseph 1965). No subcritical instabilities are possible. When the above fluid layer contains constant, distributed heat sources, Joseph & Shir (1966) have found that R_E and R_L differ by only a small amount, and that subcritical instabilities are confined to a narrow band of Rayleigh numbers beneath R_L. The addition of the constant, distributed heat sources appears in the governing equations as a real, bounded perturbation to the linear part of the system (see Davis 1969). For a sufficiently small amount of sources, the growth rates of linear theory are real (Davis 1969). The principle of exchange of stabilities is valid. Thus, the time-independent linear equations, representing the linear theory stability boundary, and the energy theory differ only by a small, bounded perturbation, and hence it is not surprising that, at least for sufficiently small amounts of heat sources, $R_E - R_L$ is small. In fact, systems whose time-independent linear operator is a small, real (non-self-adjoint) perturbation of a real, self-adjoint linear operator, and whose non-linearities are only of convective type, have the outcomes of their linear, energy and non-linear stability theories qualitatively predictable (and close). A detailed discussion of these features will be made elsewhere.

We now consider an example *related* to the above class. A thin motionless fluid layer of infinite horizontal extent is heated uniformly from below. The lower boundary is a rigid plane, and the upper is a free surface. When heated with sufficient intensity, the layer can become unstable through the joint mechanisms of buoyancy and surface tension gradients (the Bénard–Pearson problem). When gravity is absent, a linear theory has been developed by Pearson (1958), Sternling & Scriven (1964) and Smith (1966); Pearson's initial model containing a non-deflecting free surface has been improved; the new model includes the effects of the presence of capillary and gravity waves. The combined buoyancy-surface tension problem using linear theory and Pearson's model has been considered by Nield (1964). A first attempt on the surface tension problem using the methods of non-linear stability theory has been completed by Scanlon & Segel (1967), who considered an infinitely deep layer with gravity absent, and found a sub-critical instability at approximately 2·3 % below the critical Marangoni number of linear theory. In their model, it is found that there are no surface deflexions present; the free surface does not deform, and hence their analysis was based on Pearson's model.

In the present work, we will formulate the Bénard–Pearson problem for a layer whose *free surface* is *assumed* to be *non-deformable*, or in terms of non-dimensional quantities assumed to have zero Crispation number (see §2). Although this is a strong assumption, and as a result the conclusions reached here are inapplicable in various situations, the model does describe a large class of physically interesting situations (see Smith 1966), and also allows comparison with previous linear and non-linear analyses which used Pearson's model.

First, we will show that the presence of the surface tension gradients perturbs the linear operator of the Bénard problem by an unbounded operator, but one which behaves as a bounded operator. We thus expect subcritical instabilities to be confined to small band of Rayleigh numbers below the linear theory critical value, at least for sufficiently small Marangoni number.

The energy identities and consequent Euler–Lagrange equations governing the optimal stability boundary (energy theory) are derived. The Euler–Lagrange equations are shown to be merely the symmetric part of the time-independent portion of the linear theory problem. The qualitative dependence of the optimal stability boundary as a function of the parameters is examined using parametric differentiation.

The optimal stability boundary is computed and compared with Nield's (1964) linear theory and Scanlon & Segel's (1967) non-linear analysis.

2. Preliminaries

The following notation will be used: d is the mean distance between two infinite horizontal surfaces, the lower is a constant temperature, rigid plane, while the upper is a free surface on which a general heat transfer condition governs. These surfaces bound a fluid of density ρ_0. The acceleration of gravity (taken to act vertically downward) is g, and α, ν and κ are the coefficients of thermal expansion, kinematic viscosity and thermal diffusivity of the fluid, respectively. The dimensionless horizontal co-ordinates x and y, and vertical co-ordinates z, are referred to length d; the velocity vector $\mathbf{v} = (u, v, w)$, the temperature θ, the time t and the pressure p are referred to scales $\kappa/d, \Delta T, d^2/\kappa, \rho_0\kappa\nu/d^3$, where ΔT is the temperature difference across the layer. We employ the Boussinesq approximation, under which the governing equations are the following:

$$P^{-1}\frac{\partial \mathbf{v}}{\partial t} + P^{-1}\mathbf{v}.\nabla\mathbf{v} = -\nabla p + \nabla^2\mathbf{v} + R\theta\mathbf{k}, \tag{2.1a}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{v}.\nabla\theta = \nabla^2\theta + w, \tag{2.1b}$$

$$\nabla.\mathbf{v} = 0. \tag{2.1c}$$

Here, the Prandtl number $P = \nu/\kappa$, and the Rayleigh number $R = \alpha\Delta T g d^3/\kappa\nu$. The relevant boundary conditions are the following:

$$\mathbf{v} = \theta = 0, \quad \text{on} \quad z = 0, \tag{2.2a}$$

$$\theta_z + L\theta = w = u_z + B\theta_x = v_z + B\theta_y = 0, \quad \text{on} \quad z = 1. \tag{2.2b}$$

Equation (2.2*a*) expresses the facts that the lower plane is rigid and perfectly conducting. The first condition of equation (2.2*b*) is the general heat transfer relation, the second condition is the kinematic condition and the third and fourth relations express the fact that on the free surface an induced temperature gradient causes surface tractions proportional to the Marangoni number $B = s_0\Delta T d/\rho_0\nu\kappa$, where s_0 is the negative of the rate of change of the surface tension on the surface with respect to temperature. It should be emphasized that the conditions (2.2*b*) are applied on $z = 1$, only because we have *assumed* that the *free surface is non-deformable*. Although this restricts the applicability of our results to situations where the Crispation number $\rho_0\nu\kappa/sd$ is very small (see Smith 1966), a large class of physically interesting problems are covered. Here s is the mean surface tension.

We simultaneously consider two possibilities: (*a*) All dependent variables are periodic in the horizontal co-ordinates of periods $2\pi/\alpha$ and $2\pi/\beta$, respectively. The cell boundaries are assumed to have time-independent positions. The last assumption is implicitly made in all derivations of the energy identities (e.g. see Joseph 1965) for periodic disturbances. (*b*) All dependent variables are Fourier transformable in x and y.

In case (*a*), we define the integral over the volume $\mathscr{V}$ of a single cell,

$$\langle f(x,y,z,t)\rangle = \int_{x=0}^{2\pi/\alpha}\int_{y=0}^{2\pi/\beta}\int_{z=0}^{1} f(x,y,z,t)\,dz\,dy\,dx,$$

and the surface integral at $z = 1$ by

$$\int_1 f(x,y,z,t) = \int_{x=0}^{2\pi/\alpha}\int_{y=0}^{2\pi/\beta} f(x,y,1,t)\,dy\,dx.$$

In case (*b*), the x and y limits in the above integrals are taken from $-\infty$ to ∞.

Let us consider the space $\mathscr{S}$ of seven vectors Ψ,

$$\mathscr{S} = \Big\{\Psi \mid \Psi = (\mathbf{v}, \theta, u(x,y,1,t),\quad v(x,y,1,t),\quad \theta(x,y,1,t)),$$

$\nabla . \mathbf{v} = 0,\quad \mathbf{v}(x,y,0,t) = \theta(x,y,0,t) = w(x,y,1,t) = 0;$
$(\mathbf{v},\theta)$ has continuous second partial derivatives; Ψ is either periodic in x and y of period $2\pi/\alpha$ and $2\pi/\beta$ respectively, or else Fourier transformable in x and y; the scalar product of Ψ_1 and Ψ_2 is defined by

$$\langle \mathbf{v}_1 . \mathbf{v}_2 + \lambda\theta_1\theta_2\rangle + \int_1 (u_1u_2 + v_1v_2 + \lambda\theta_1\theta_2),\quad \lambda > 0\Big\}.$$

We have generalized the scalar product by the introduction of the positive parameter λ, in order to obtain the analogy with the 'linking' parameter defined by Joseph (1965). The time-independent part of the linearized version of system (2.1) can then be written as

$$\bar{L}\Psi - P\Psi = 0,$$

where

$$P\Psi = \begin{bmatrix} \nabla p \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\bar{L} = \begin{bmatrix} A_{11} & O_{43} \\ O_{34} & A_2 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} \nabla^2 & 0 & 0 & 0 \\ 0 & \nabla^2 & 0 & 0 \\ 0 & 0 & \nabla^2 & R \\ 0 & 0 & 1 & \nabla^2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -\frac{\partial}{\partial z} & 0 & -B\frac{\partial}{\partial x} \\ 0 & -\frac{\partial}{\partial z} & -B\frac{\partial}{\partial y} \\ 0 & 0 & -\frac{\partial}{\partial z} - L \end{bmatrix},$$

and O_{ij} is the $i \times j$ zero matrix.

(i) *Demonstration that the surface tension terms in the periodic case behave as a bounded perturbation to the Bénard problem*

We wish to regard the operator $\bar{L}$ as consisting of two parts. The first is the self-adjoint operator corresponding to only buoyancy driven connection ($B = 0$) plus a second part M, which denotes the contribution due to surface tension gradients. If we regard B as small, and the surface tension contributions as perturbations on the Bénard convection problem, the perturbation operator has the form

$$M = \begin{bmatrix} O_{44} & O_{43} \\ O_{34} & C \end{bmatrix},$$

where
$$C = -\frac{B}{\sqrt{R}}\begin{bmatrix} 0 & 0 & \dfrac{\partial}{\partial x} \\ 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & 0 & 0 \end{bmatrix}.$$

For convenience of the demonstration we have let $\phi = \sqrt{R}\,\theta$ and $\lambda = R$. The norm $\|C\|$ of C can be found for

$$\int_1 (u^2+v^2+\phi^2) = 1$$

as the supremum of
$$\frac{B}{\sqrt{R}}\left|\int_1 (u\phi_x + v\phi_y)\right|. \tag{2.3a}$$

But
$$|u|\,|\phi_x| + |v|\,|\phi_y| \leqslant \tfrac{1}{2}(u^2+\phi_x^2+v^2+\phi_y^2). \tag{2.3b}$$

In the case of *periodic disturbances* of overall wave-number a, $a^2, = \alpha^2+\beta^2$, linear theory allows consideration of disturbances of the form $\cos\alpha x\cos\beta y$. We wish to estimate the effect of C in the linear theory, so that we also assume this form of disturbance. It follows that

$$\int_1 (\phi_x^2+\phi_y^2) = a^2\int_1 \phi^2. \tag{2.4}$$

Using both (2.3) and (2.4), we find that

$$\frac{B}{\sqrt{R}}\left|\int_1 (u\phi_x+v\phi_y)\right| \leqslant \frac{1}{2}\frac{B}{\sqrt{R}}\int_1 (u^2+v^2+a^2\phi^2) \leqslant \frac{1}{2}\frac{B}{\sqrt{R}}\max(a^2, 1),$$

so that
$$\|M\| = \|C\| \leqslant \tfrac{1}{2}(B/\sqrt{R})\max(a^2,1) \quad \text{valid for} \quad R > 0.$$

We have tried to estimate that part of the time-independent *linear operator* containing surface tension contributions. According to *linear theory*, as $a^2 \to \infty$, $R_L \sim a^4$. As $a^2 \to 0$, $R_L \sim a^{-2}$. Thus, $\frac{1}{2}BR^{-\frac{1}{2}}\max(a^2, 1)$ is always bounded, and is certainly small when B is small. Furthermore, it turns out that in the forthcoming energy theory, the preferred value of a^2 changes very little, so that $\|C\|$ is small in the energy theory as well. The fact that $\|C\|$ is bounded by a^2, for $a^2 > 1$, reflects the fact that differential operators are generally unbounded. By restricting ourselves to a finite interval of wave-numbers, due to a physical preference, we see that C behaves as would a bounded perturbation operator, and so we might expect the linear theory and energy limit to be rather close, at least for sufficiently small B.

It will be convenient later to have the symmetric part of $\bar{L}$. We again consider the original system $\bar{L}\Psi - P\Psi = 0$ with $\Psi \in \mathscr{S}$ (i.e. with θ replacing $R^{-\frac{1}{2}}\phi$ and Fourier transformable as well as periodic disturbances) and with arbitrary λ, $\lambda > 0$. By standard methods we find that the adjoint problem is

$$\tilde{L}\tilde{\Psi} - P\tilde{\Psi} = 0,$$

where
$$\tilde{L} = \begin{bmatrix} A_{12} & O_{43} \\ O_{34} & A_3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -\dfrac{\partial}{\partial z} & 0 & 0 \\ 0 & -\dfrac{\partial}{\partial z} & 0 \\ \dfrac{B}{\lambda}\dfrac{\partial}{\partial x} & \dfrac{B}{\lambda}\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial z}-L \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \nabla^2 & 0 & 0 & 0 \\ 0 & \nabla^2 & 0 & 0 \\ 0 & 0 & \nabla^2 & \lambda \\ 0 & 0 & R/\lambda & \nabla^2 \end{bmatrix}.$$

Since both $\tilde{L}$ and L are defined on the *same space* $\mathscr{S}$, we can write

$$\bar{L} = \tfrac{1}{2}(\bar{L}+\tilde{L})+\tfrac{1}{2}(\bar{L}-\tilde{L}).$$

The symmetric part L_s is

$$L_s = \begin{bmatrix} A_{13} & O_{43} \\ O_{34} & A_4 \end{bmatrix}, \tag{2.5}$$

where

$$A_4 = \begin{bmatrix} -\dfrac{\partial}{\partial z} & 0 & -\tfrac{1}{2}B\dfrac{\partial}{\partial x} \\ 0 & -\dfrac{\partial}{\partial z} & -\tfrac{1}{2}B\dfrac{\partial}{\partial y} \\ \dfrac{1}{2}\dfrac{B}{\lambda}\dfrac{\partial}{\partial x} & \dfrac{1}{2}\dfrac{B}{\lambda}\dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial z}-L \end{bmatrix}, \quad A_{13} = \begin{bmatrix} \nabla^2 & 0 & 0 & 0 \\ 0 & \nabla^2 & 0 & 0 \\ 0 & 0 & \nabla^2 & \tfrac{1}{2}(R+\lambda) \\ 0 & 0 & \dfrac{1}{2}\dfrac{(R+\lambda)}{\lambda} & \nabla^2 \end{bmatrix}.$$

(ii) *The energy identities*

Let us define

$$\mathscr{D}(\theta) = \langle \nabla\theta . \nabla\theta \rangle + L\int_1 \theta^2, \tag{2.6a}$$

$$D(\mathbf{v}) = \langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle. \tag{2.6b}$$

Let us dot (2.1a) with $\mathbf{v}$ and integrate over $\mathscr{V}$. We obtain, using boundary conditions (2.2) and definition (2.6b),

$$\frac{1}{2}\frac{d}{dt}\langle P^{-1}\mathbf{v}.\mathbf{v}\rangle = -B\int_1 \theta w_z + R\langle w\theta\rangle - D(\mathbf{v}). \tag{2.7a}$$

Let us multiply (2.1b) by θ, and integrate over $\mathscr{V}$. We obtain, using boundary conditions (2.2) and definition (2.6a),

$$\frac{1}{2}\frac{d}{dt}\langle \theta^2\rangle = \langle w\theta\rangle - \mathscr{D}(\theta). \tag{2.7b}$$

We form the sum of (2.7a) and λ, $\lambda > 0$ times (2.7b), and obtain

$$\frac{1}{2}\frac{d}{dt}\langle P^{-1}\mathbf{v}.\mathbf{v}+\lambda\theta^2\rangle = (R+\lambda)\langle w\theta\rangle - [D(\mathbf{v})+\lambda\mathscr{D}(\theta)] - B\int_1 \theta w_z. \tag{2.8}$$

In order to symmetrize the problem, let us make the following changes of variable:

$$\phi = \sqrt{\lambda}\,\theta, \quad \lambda = B\mu, \quad R = BN_r, \tag{2.9}$$

where B and R are taken to be positive. Equation (2.8) then takes the form,

$$[D(\mathbf{v})+\mathscr{D}(\phi)]^{-1}\frac{dE}{dt} = \left\{-1+\sqrt{\frac{B}{\mu}}[D(\mathbf{v})+\mathscr{D}(\phi)]^{-1}\left[(\mu+N_r)\langle w\phi\rangle - \int_1 \phi w_z\right]\right\}, \tag{2.10}$$

where $E = \frac{1}{2}\langle P^{-1}\mathbf{v}.\mathbf{v}+\phi^2\rangle$ is an energy functional which is positive whenever λ is positive. It follows from (2.10) that

$$[D(\mathbf{v})+\mathscr{D}(\phi)]^{-1}\frac{dE}{dt} \leqslant -1+\frac{\sqrt{B}}{\rho}, \tag{2.11}$$

whenever the following maximum problem is satisfied:

(iii) *The maximum problem*

$$\rho^{-1} = \max_{\mathscr{T}}\left\{\frac{\mu+N_r}{\sqrt{\mu}}\langle w\phi\rangle+\frac{1}{\sqrt{\mu}}\int_1(-\phi w_z)\right\}, \tag{M}$$

and

$$D(\mathbf{v})+\mathscr{D}(\phi) = 1.$$

Here $\mathscr{T} = \{(\mathbf{v},\phi)\,|\,(\mathbf{v},\phi)$ have continuous second partial derivatives, $\nabla.\mathbf{v} = 0$, $\mathbf{v}(x,y,0,t) = \phi(x,y,0,t) = w(x,y,1,t) = 0$, and either $(\mathbf{v},\phi)$ is periodic in x and y of period, say, $2\pi/\alpha$ and $2\pi/\beta$ respectively, or $(\mathbf{v},\phi)$ is Fourier transformable in x and $y\}$.

Let the inequalities

$$D(\mathbf{v}) \geqslant \tfrac{1}{2}a_1^2P^{-1}\langle\mathbf{v}.\mathbf{v}\rangle,$$
$$\mathscr{D}(\phi) \geqslant \tfrac{1}{2}a_2^2\langle\phi^2\rangle,$$

with $a_1^2 > 0$ and $a_2^2 > 0$, hold. Then, for any fixed values of $\mu > 0$ and $N_r \geqslant 0$, $\sqrt{B} < \rho(\mu, N_r)$, in the time interval $[0,\tau]$, we have

$$E(\tau) \leqslant E(0)\exp\left\{-\left(1-\frac{\sqrt{B}}{\rho}\right)\xi^2\tau\right\},$$

where $E(0)$ is the initial energy of the disturbance and $\xi^2 = \min(a_1^2, a_2^2)$. If $\sqrt{B} < \rho$ for all τ, then $E \to 0$, and the flow is asymptotically stable in the mean. The proof follows directly (Joseph & Shir 1966). We thus have certain asymptotic stability if $\sqrt{B} < \rho$.

We can obtain the Euler–Lagrange equations for the maximum problem by introducing Lagrange multipliers B_μ and $\Pi(x,y,z,t)$. The equations result from

$$\delta\left\{\frac{\mu+N_r}{\sqrt{\mu}}\langle w\phi\rangle-\frac{1}{\sqrt{\mu}}\int_1\phi w_z+\left\langle\frac{2\Pi}{B_\mu}\nabla.\mathbf{v}\right\rangle-\frac{1}{B_\mu}[D(\mathbf{v})+\mathscr{D}(\phi)]\right\} = 0.$$

They are:

$$\left.\begin{aligned}&\nabla^2\phi+\tfrac{1}{2}B_\mu\frac{\mu+N_r}{\sqrt{\mu}}w = 0,\\ &\nabla^2\mathbf{v}+\tfrac{1}{2}B_\mu\frac{\mu+N_r}{\sqrt{\mu}}\phi\mathbf{k}-\nabla\Pi = 0,\\ &\nabla.\mathbf{v} = 0,\end{aligned}\right\} \tag{2.12}$$

with the boundary conditions

$$\mathbf{v} = \phi = 0, \quad z = 0, \tag{2.13a}$$

$$\phi_z+L\phi+\frac{1}{2}\frac{B_\mu}{\sqrt{\mu}}w_z = w = u_z+\frac{1}{2}\frac{B_\mu}{\sqrt{\mu}}\phi_x = v_z+\frac{1}{2}\frac{B_\mu}{\sqrt{\mu}}\phi_y = 0, \quad z = 1. \tag{2.13b}$$

The last three conditions of (2.13*b*) are 'natural' boundary conditions (see Courant & Hilbert 1961, p. 208), often called the dynamic boundary conditions. For any solution of the above, we have that

$$\langle \phi \nabla^2 \phi \rangle = -\mathscr{D}(\phi) - \frac{1}{2}\frac{B_\mu}{\sqrt{\mu}}\int_1 \phi w_z,$$

$$\langle \mathbf{v}.\nabla^2 \mathbf{v} \rangle = -D(\mathbf{v}) - \frac{1}{2}\frac{B_\mu}{\sqrt{\mu}}\int_1 \phi w_z,$$

so that we can obtain

$$-[D(\mathbf{v}) + \mathscr{D}(\phi)] - \frac{B_\mu}{\sqrt{\mu}}\int_1 \phi w_z + B_\mu \frac{\mu + N_r}{\sqrt{\mu}}\langle \phi w \rangle = 0.$$

Since

$$D(\mathbf{v}) + \mathscr{D}(\phi) = 1,$$

$$\frac{1}{B_\mu} = \frac{\mu + N_r}{\sqrt{\mu}}\langle \phi w \rangle - \frac{1}{\sqrt{\mu}}\int_1 \phi w_z. \tag{2.14}$$

Thus, the minimum of the positive $\{B_\mu\}$ coincides with the solution of the maximum problem, i.e.

$$\rho(\mu, N_r) = \min B_\mu(N_r),$$

for any of the positive set of eigenvalues $\{B_\mu\}$. The domain of stability is largest when we choose that value of μ, μ_c, giving the maximum value of ρ. Thus,

$$B_E^{\frac{1}{2}}(N_r) = \max_{\mu > 0} \rho(\mu, N_r) = \max_{\mu > 0} [\min B_\mu(N_r)]. \tag{2.15}$$

We note that the *Euler–Lagrange system* (2.12) and (2.13) is *identical* to the *symmetric part* (2.5) of the *linear time-dependent operator* involved in the governing equations. In addition to (2.5) being more easily obtained than the system (2.12) and (2.13), the analysis shows us the relationship between the equations which govern the energy theory and the linear theory. When the time-independent part of the linear theory equation is only a small perturbation from a self-adjoint system, the eigenvalues of its symmetric part (the energy problem) cannot be vastly different.

3. Parametric differentiation

Let $B_\mu(\mu, N_r)$ and $(\mathbf{v}(x, y, z; \mu, N_r), \phi(x, y, z; \mu, N_r))$, which solve the maximum problem (M) and the equivalent Euler–Lagrange system (2.12), (2.13), be continuously differentiable functions of their arguments. Let the best value μ_c of the coupling parameter μ be finite and non-zero. Then

$$\mu_c = N_r + \int_1 (-\phi w_z)/\langle w\phi \rangle. \tag{3.1}$$

Proof

Let $(\mathbf{v}, \phi)$ be a solution for any fixed values of μ and N_r. Consider two such solutions, and label them with subscripts. From (2.12) and (2.13) we can find:

$$(a) \quad -L_1 \int_1 \phi_1 \phi_2 - \frac{1}{2}\frac{B_\mu^{(1)}}{\sqrt{\mu_1}}\int_1 \phi_2 w_{1z} - \langle \nabla\phi_1 . \nabla\phi_2 \rangle + \tfrac{1}{2}B_\mu^{(1)} \frac{\mu_1 + N_r^{(1)}}{\sqrt{\mu_1}}\langle w_1 \phi_2 \rangle = 0,$$

$$(b)\quad -L_2\int_1 \phi_1\phi_2-\frac{1}{2}\frac{B_\mu^{(2)}}{\sqrt{\mu_2}}\int_1 \phi_1 w_{2z}-\langle\nabla\phi_1.\nabla\phi_2\rangle+\tfrac{1}{2}B_\mu^{(2)}\frac{\mu_2+N_r^{(2)}}{\sqrt{\mu_2}}\langle w_2\phi_1\rangle = 0,$$

$$(c)\quad -\frac{1}{2}\frac{B_\mu^{(1)}}{\sqrt{\mu_1}}\int_1 \phi_1 w_{2z}-\langle\nabla\mathbf{v}_1:\nabla\mathbf{v}_2\rangle+\tfrac{1}{2}B_\mu^{(1)}\frac{\mu_1+N_r^{(1)}}{\sqrt{\mu_1}}\langle\phi_1 w_2\rangle = 0,$$

$$(d)\quad -\frac{1}{2}\frac{B_\mu^{(2)}}{\sqrt{\mu_2}}\int_1 \phi_2 w_{1z}-\langle\nabla\mathbf{v}_1:\nabla\mathbf{v}_2\rangle+\tfrac{1}{2}B_\mu^{(2)}\frac{\mu_2+N_r^{(2)}}{\sqrt{\mu_2}}\langle\phi_2 w_1\rangle = 0.$$

If we form the sum of $(a)-(b)+(c)-(d)$ and allow the solutions to coalesce, we obtain the general relation among solutions as the parameters are allowed to vary:

$$-\delta L\int_1 \phi^2-\tfrac{1}{2}\delta\left(\frac{B_\mu}{\sqrt{\mu}}\right)\int_1 \phi w_z+\tfrac{1}{2}\delta\left(B_\mu\frac{\mu+N_r}{\sqrt{\mu}}\right)\langle w\phi\rangle = 0. \qquad (3.2)$$

For fixed N_r and L, we find that

$$\frac{\partial B_\mu}{\partial\mu}=\frac{1}{2}\frac{B_\mu}{\mu}\left\{\frac{1-(\mu-N_r)\langle w\phi\rangle\Big/\int_1(-\phi w_z)}{1+(\mu+N_r)\langle w\phi\rangle\Big/\int_1(-\phi w_z)}\right\}.$$

But, subject to our hypotheses, the optimum value of μ is determined by the relation

$$\frac{\partial B_\mu}{\partial\mu}=0,$$

so that

$$\mu_c = N_r+\int_1(-\phi w_z)/\langle w\phi\rangle.$$

It can be shown that, since $\mu>0$, μ_c corresponds to a maximum of B_μ.

The integral $\int_1(-\phi w_z)$ gives a measure of two features of the free surface. The surface tension mechanism is made effective by temperature gradients along the free surface. Thus, if $\phi(x,y,1,t)$ were zero, there would be no surface tension gradient. The rigidity of the free surface is measured by w_z. When

$$u(x,y,1,t)=v(x,y,1,t)=0,\quad w_z(x,y,1,t)=0$$

and there is no slip on $z=1$.

The expression (3.1) for μ_c is useful in computing the optimal stability boundary since for $B=0$, $\lambda_c=B\mu_c$ is merely R_L and λ_c increases monotonically with increasing B.

The optimal stability boundary $B_E^{\frac{1}{2}}(N_r)$ is a *monotone decreasing function* of N_r. For, from (3.2), for fixed μ and L, we have

$$\frac{\partial B_\mu}{\partial N_r}=-\frac{B_\mu}{\int_1(-\phi w_z)+(\mu+N_r)\langle w\phi\rangle},$$

and, from (2.14), we have

$$\frac{\partial B_\mu}{\partial N_r}=-\frac{B_\mu^2}{\sqrt{\mu}}<0.$$

The *values* of $B_{\mu}^{\frac{1}{2}}(N_r)$ *increase monotonically* with L. For, from (3.2), for fixed μ and N_r, we have

$$\frac{\partial B_\mu}{\partial L} = \frac{2\sqrt{\mu}\int_1 \phi^2}{\int_1 (-\phi w_z) + (\mu + N_r)\langle w\phi\rangle} = 2B_\mu \int_1 \phi^2 > 0,$$

where we have used (2.14).

4. Computation of the optimal stability boundary

Our scheme for the computation of $B_E(N_r)$ is as follows:

(*a*) Separate variables or Fourier transform system (2.12) and (2.13). In either case, the only influence of the x and y variations is to introduce an overall wave-number $a > 0$. (*b*) For fixed values of N_r, L, μ find $\min_a B_\mu$. This was done numerically by the Runge–Kutta–Gill method. This was found to be both faster

R	B_L	B_E
	$L = 0$	
0·0	79·61	56·77
100·0	68·43	50·59
200·0	57·12	43·87
300·0	45·49	36·51
400·0	33·59	28·32
500·0	21·39	19·07
600·0	8·857	8·423
669·0	0·000	0·000
	$L = 10$	
0·0	413·4	180·7
100·0	378·7	171·3
300·0	305·0	150·2
500·0	225·1	124·3
700·0	138·6	89·99
900·0	44·73	37·39
989·49	0·000	0·000
	$L = 1000$	
0·0	3217×10	—
200·0	2752×10	1957
400·0	2238×10	1783
600·0	1671×10	1576
800·0	1049×10	1277
1000·0	3647	769·1
1099·12	0·000	0·000

TABLE 1. The critical Marangoni numbers B_E and B_L for the energy theory and linear theory are given for various Rayleigh numbers R and for $L = 0$, 10, 1000 ($L = 0$ corresponds to an insulating upper free surface, $L = \infty$ corresponds to a perfectly conducting upper free surface)

and more accurate than an 'exact' solution by Fourier series. (*c*) The value of μ was varied to give a maximum value of B_{μ}. This was called $B_E^{\frac{1}{2}}(N_r)$; this locus is the optimal stability boundary for the given value of L.

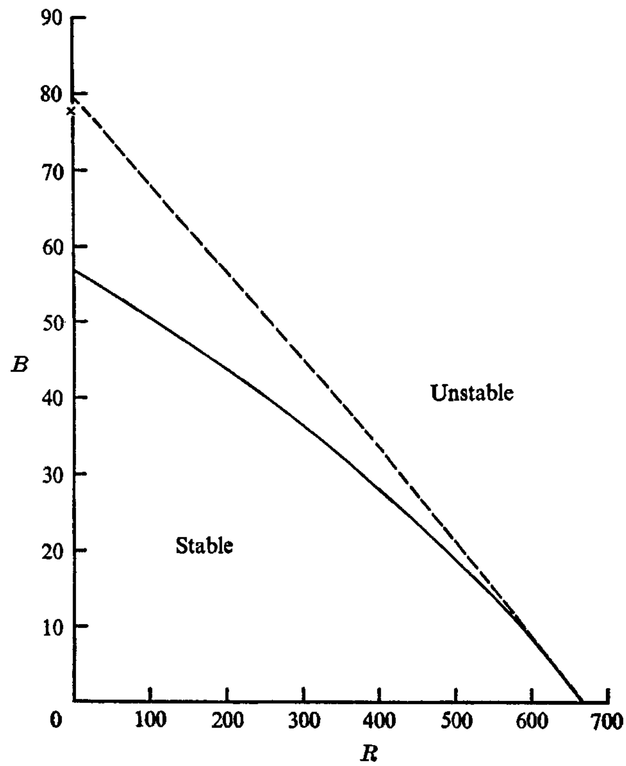

FIGURE 1. Stability curves: $L = 0$, B *vs.* R, ——— linear theory; ——, energy theory.

Table 1 presents the relevant numerical results. Both B_E and B_L, the linear theory critical Marangoni number, are presented for various values of L. Figures 1–3 illustrate the results. For $L = 0$, subcritical instabilities are allowable at $R = 0$ (the Pearson problem) for $56{\cdot}77 \leqslant B < 79{\cdot}61$, or within $28{\cdot}7$ % of the linear theory prediction. This band of allowable subcritical instabilities shrinks to zero as $R \to 669{\cdot}0$, at which point $B = 0$. This is the Bénard problem, and the result is in accord with Joseph (1965).

For small values of B, we can estimate the norm of the surface tension terms, $\frac{1}{2}B/\sqrt{R}\max(a^2, 1)$ by using the approximate linear theory result given by Nield (1964), $R/R_c + B/B_c = 1$, where $R = R_c$ when $B = 0$, and $B = B_c$ when $R = 0$. We find that, as L increases, both B and R increase, so that $B/\sqrt{R}$ increases with increasing L (for a^2 slowly varying). Thus, the norm of the surface tension terms increases with increasing L; we therefore expect the results of the energy and linear theories to differ more for large L than for small L. This is borne out by the numerical results in figures 1–3.

Scanlon & Segel (1967) have used the methods of non-linear hydrodynamic stability on the surface tension problem with no gravity ($R = 0$). They considered

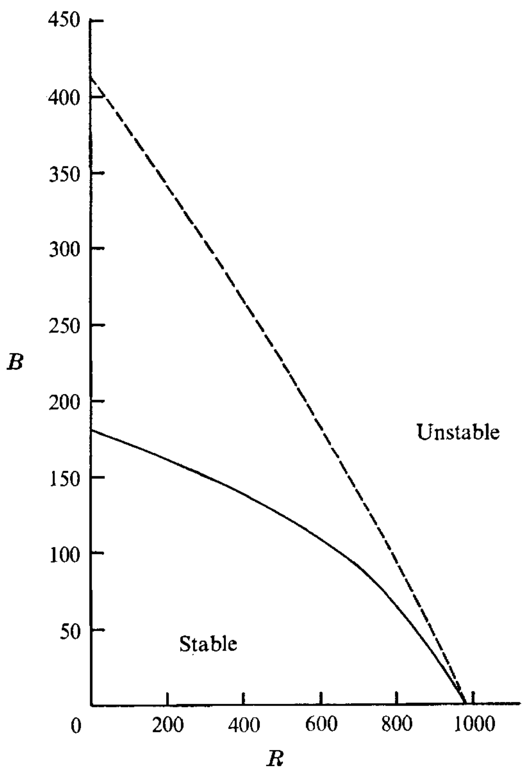

FIGURE 2. Stability curves: $L = 10$, B *vs.* R, ——— linear theory; ——, energy theory.

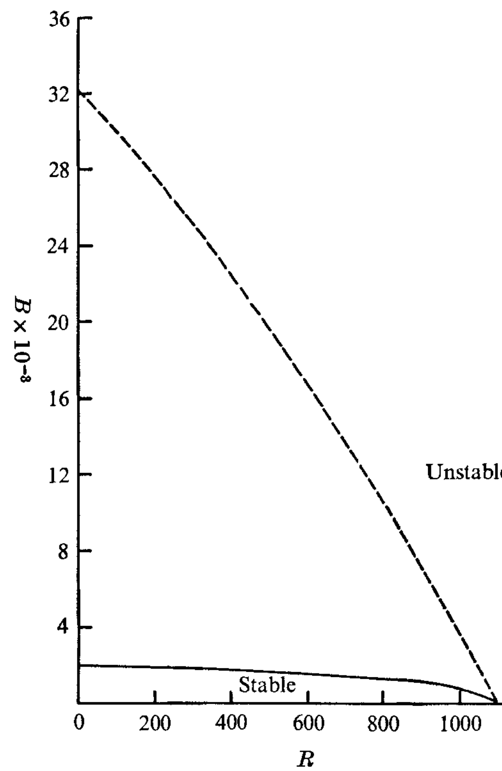

FIGURE 3. Stability curves: $L = 1000$, B *v*
——— linear theory; ——, energy theo

infinite Prandtl number, $L = 0$ and a layer of infinite depth. In this problem, surface deflexions are identically zero at each order, so that Pearson's (1958) model is valid. They found that subcritical instabilities were possible 2·3 % below B_L, the Marangoni number of linear theory. That result, scaled as $B_E = 0{\cdot}977 B_L$, is marked with a cross on figure 1, and is seen to be well within the allowable band of subcritical instabilities demarked herein.

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